Recursive Equations and the Generating Function

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1 Introduction

1.1 What is a recersive function?

A recursive function is a function in which each element of the sequence is a function of the preceding element.

More formally: Given a sequence $\{x_n\}$, if there exist a positive integer k and a equation that connects x_{n+k} and the k terms before $x_{n+k-1}, x_{n+k-2}, ..., x_n$ i.e if there exist a $k \in \mathbb{N}^+$ and an equation

$$\Phi(x_{n+k}, x_{n+k-1}, x_{n+k-2}, ..., x_n) = 0, k = 0, 1, 2, ... \tag{1}$$

then we call the sequence $\{x_n\}$ a k-order recursive sequence, and call Equation 1 the recursive equation of sequence $\{x_n\}$.

From Equation 1 we also have

$$x_{n+k} = \phi(x_{n+k-1}, x_{n+k-2}, ..., x_n) \tag{2}$$

which is called the recursive formula of sequence $\{x_n\}$.
The values of the first k terms from the beginning of the sequence is called the initial value or initial condition of the recursive equation/formula. A k-order recursive sequence is uniquely determined by the recursive formula and the k initial values.
In math olympiads, we usually don't have problems where the function is too complicated, it's usually just a linear function with constant coefficients, see below:
The sequence determined by initial k values, and recursive formula, (is constant and) is called k-order constant coefficient linear recursion sequence, and in particular if , we call it k-order constant coefficient linear homogeneous recursion sequence, or k-order linear homogeneous recursive sequence in short.

1.2 What is a generating function?

Given a n-degree polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n \tag{3}$$

if we extract out the coefficients a_i(i=0,1,...,n), we would get a finite sequence:

$$a_0, a_1, a_2, ..., a_n \tag{4}$$

On the other hand, given a finite sequence (Equation 4), and letting it be the coefficients of a one variable polynomial, a determined n-degree polynomial (Equation 3) will be obtained.
In general, the polynomial (Equation 3) is called the generated function of sequence (4).
If we can express the generated function $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ in a simple expression, then by taking x to be some specific values on purpose, would then yield special relations about the sequence $a_0, a_1, a_2, ..., a_n$.
For example, in the identity

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n}x^n$$

we can let x take some specific values (1,-1 etc. . .). We would then get a particular combinatorial identity. Therefore, generating function is strongly related to combinatorial identity. The same is also true for recursive functions, but it will not be covered here.

For an infinite sequence, we can define the generated function analogously, according to the definition of the generated function of a finite sequence.
Thus, the generated function of an infinite sequence:

$$a_0, a_1, a_2, ..., a_n, ...$$

should be an "infinite degree polynomial":

$$a_0 + a_1x + a_2x^2 + ... + a_nx^n + ... \tag{5}$$

For the convenience of discussing the problems, we call this "infinite degree polynomial" a formal power series. This name can be understood as follows: the formula of adding infinite numbers is called a series, and each term in (Equation 5) is a power series $a_n x^n$, so it is called a power series.

We have in general

Definition 1: Given a infinite sequence

$$a_0, a_1, a_2, ..., a_n, ...$$

we call the formal power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ... = \sum_{n=0}^{\infty} a_n x^n$$

its generated function.

Definition 2: Given two formal power series and . If and only if , k=0,1,2,..., then we call them equal or identical, and denote it by .

2 Basic knowledge of recersive/generating functions

2.1 The Solution of General Term Formula of Typical Recursive Equation

Now that we have some idea of what a recursive equation is, we naturally hope to solve it like how we solved equations in junior school, just that the variables are different, i.e we want to find the general term of a sequence x_n given relation between the related terms, insead of to find unknown real numbers x,y,z given their quantitative relation.

This part will show how to solve the following common recurrance equations:
Arithmetic sequence: $x_n = x_{n-1} + d$
Geometric sequence: $x_n = r x_{n-1}$
Second-order linear homogeneous recursive sequence: $x_n = a x_{n-1} + b x_{n-2}$
k-order linear homogeneous recursive sequence: $x_n = a_1 x_{n-1} + a_2 x_{n-2} + ... + a_k x_{n-k}$
Fractional linear recursive sequence: $x_n = \frac{a x_{n-1} + b}{c x_{n-1} + d}$

and various recursive equations that can be reduced to above equations...

When we are solving recurrence equations, we should keep in mind three methods: reduction, characteristic equation, and fixed point. You'll understand what they mean as we go on.
In high school, we have learned the general formula for arithmetic/geometric sequence already, and we will see how we can magically reduce a recursive equation into them in a few moments.
But first of all, what is reduction?
Reduction is a kind of thinking of transformation, which refers to a way of thinking that reduces the problem to be solved to a problem that can be solved within the scope of existing knowledge through transformation. Reduction is the most widely used way of thinking in mathematics. It can be said that solving mathematical problems is a transformation problem, and every mathematical problem is solved through continuous transformation. One can say that reduction is the direction of problem solving.
The so-called reduction generally transforms the abstract into the concrete, the complex into the simple, the unknown into the known, and the unfamiliar into the familiar. And the biggest reduction is no more than to reduce the unresolved problems to the solved problems.
Actually, the idea of the recursive equation itself is a reduction –Reducing the general terms into the preceding elements, then again to the initial values. But what we are talking about here specifically, is to transform an unknown recursive structure into a known structure, so that we can leverage the ladder general term formula to deduce the original one.

Example 1:

$$x_n = p x_{n-1} + q \tag{6}$$

This is a first-order linear (inhomogeneous) recursive sequence, which we do not emphasize because it is trivial in nature.

Method a)
Construct λ so that Equation 6 transforms into $x_n + \lambda = p(x_{n-1} + \lambda)$, where it suffices take λ to be $\frac{q}{p-1}$ (it's easy to verify), thus $\{x_n + \lambda\}$ becomes a geometric sequence, which is trivial since we can find the general formula for it then subtract by λ to find x_n.

Method b)
Divide Equation 6 by p^n on both sides so that it transforms into $\frac{x_n}{p^n} = \frac{x_{n-1}}{p^{n-1}} + \frac{q}{p^n}$, if we define a new sequence $y_n = \frac{x_n}{p^n}$ we will see that it has a known structure that is discussed in a later section, where we talk about the Cumulative addition method, so the problem is again solved.

In essence, both methods are motivated by observing the structure, and then transforming the algebraic expression to make the subscripts match, it is converted into a familiar structure, and finally counted by cumulative method (see the section below where I share three more methods) or some known formula.
Now, let us look at a somewhat more complex reduction.

Example 2:

$$x_n = \frac{ax_{n-1} + b}{cx_{n-1} + d} \tag{7}$$

This is a fractional linear recursive sequence as we already mentioned, that we can solve using fixed points. If we ignore the fraction and only look at it locally, we will see it has the same structure as example 1, and yes we are again reducing it to arithmetic/geometric sequence!

Method
Consider the fix points of the equation y_1,y_2, i.e the solution of equation $x = \frac{ax+b}{cx+d}$.
(Note that we didn't use x_1,x_2 since it would confuse with the sequence $\{x_n\}$)
If $y_1 = y_2$, then we have λ such that $\frac{1}{x_n - y_1} = \frac{1}{x_{n-1} - y_1} + \lambda$, where λ can be easily reverse engineered by balancing the coefficients. But anyways, we see that $\{\frac{1}{x_n - y_1}\}$ becomes an arithmetic sequence, the rest is trivial.
If $y_1 \neq y_2$, then we have $\frac{x_n - y_1}{x_n - y_2} = \lambda \frac{x_{n-1} - y_1}{x_{n-1} - y_2}$ for some λ that can be easily figured out.
Thus the transformation $\{\frac{x_n - y_1}{x_n - y_2}\}$ makes the new sequence a geometric sequence, which again is trivial.
We are only illustrating the fact that should be remembered, the long and boring calculations are not suitable here since it would ruin the flow. But the reader can verify the facts himself.

Example 3:

$$x_n = ax_{n-1} + bx_{n-2} \tag{8}$$

This is a second-order linear homogeneous recursive sequence, and sequences of k-order in general have similar solutions, so that we are only presenting solution to second-order recursive sequence, since it is more useful, and notice that this can be directly applied to rewrite fibonacci sequence in general term formula.

Method
Consider the solutions y_1,y_2 of the characteristic equation of Equation 8, i.e the solutions/roots of equation $x^2 = ax + b$
If $y_1 = y_2$, then $x_n = (A + Bn)y_1^n$,
where A and B can be reverse engineered through balancing coefficients, i.e comparing the coefficients of equation gained by the initial values x_1,x_2 this ensures two equations so that we can solve out A and B, and sometimes we can define x_0 and combine with x_1 to make this easier.
Also, sometimes we only have the recursive equation while the initial values are to be determined, then we can choose A and B for our own purpose, for example if we want all terms of the sequence to be congruent with some value (mod p), then we could choose A and B accordingly.
If $y_1 \neq y_2$, then $x_n = Ay_1^n + By_2^n$
where A and B can be determined by the initial values.
Above two cases both can be easily verified that it is indeed solution for the recursive equation, since y_1,y_2 are designed to satisfy some certain property that ensures the solution. But simply verifying the step-by-step logical derivation of a proof without trying to gain insight into the ideas behind this series of derivations does not count as understanding the proof. The reader should try to

reverse engineer the property we designed y_1,y_2 with and see how and why it relates to the solution.

Example 4:

$$x_n = ax_{n-1} + bx_{n-2} + c \tag{9}$$

This looks similar to example 3 doesn't it? The difference between this and example 3 is just that there is a constant c, making the equation inhomogeneous. But it can be reduced into example 3 by constructing λ such that

$$x_n + \lambda = a(x_{n-1} + \lambda) + b(x_{n-2} + \lambda) \tag{10}$$

in other words, we "stuffed in" the constant c into the terms, with the same proportion, and thus transforms/reduces the structure of the equation to the structure of equation in example 3. Here, can be easily calculated out, and the rest is trivial.

2.2 Three More Methods to Solve Recursive Equations

2.2.1 Substitution Method

The main idea of this method is to choose an appropriate function $\phi(x)$, and let $x_n = \phi(y_n)$ or vice versa, and substitute it into the recursive relation of $\{x_n\}$, to get a new recursive relation about $\{y_n\}$. If we can find out the general term formula of $\{y_n\}$, then substituting it back into $x_n = \phi(y_n)$ or the other way around will generate the general formula for $\{x_n\}$. Therefore, the key of this method is the choice of function/transformation $\phi(x)$.
If the above seems familiar, then yes, we have used the idea already when we reduced some unfamiliar sequences to arithmetic/geometric or other sequences that we already know.

2.2.2 Cumulative method

For sequences of structure $x_n = x_{n-1} + f(n-1)$, we can according to the recursive relation, write down all equations of n where n traverse through $1 \to n$, then add them up on both sides separately to get the general term formula.

Example:

$$x_n = x_{n-1} + f(n-1) \tag{11}$$

Solution: Construct

$$x_n - x_{n-1} = f(n-1)$$
$$x_{n-1} - x_{n-2} = f(n-2)$$
$$...$$
$$x_2 - x_1 = f(1)$$

The final result of $\{x_n\}$ can be obtained by adding both sides of the above n-1 expressions separately. The rest is trivial.

Comment:
1. If f(n) is a first-order function, then the sum up can be converted into a sum up of the arithmetic sequence.
2. If f(n) is an exponential function, then the sum up can be converted into a sum up of a geometric sequence.
3. If f(n) is a second-order function, then the sum up can be computed through regrouping of the terms.
4. If f(n) is a fractional function, then the sum up can be computed through telescoping of the terms.

Above method is called the cumulative method, but more specifically, the cumulative addition method.

We also have the cumulative multiplication method.

The cumulative multiplication method is to find the formula of the general term of the sequence by multiplying each level step by step and eliminating/canceling out terms of the sequence. Sequences of structure $x_n = x_{n-1}f(n-1)$ can be solved by this method,
More specifically, we construct

$$\frac{x_n}{x_{n-1}} = f(n-1)$$
$$\frac{x_{n-1}}{x_{n-2}} = f(n-2)$$
$$...$$
$$\frac{x_2}{x_1} = f(1)$$

and the rest is trivial.

Comment:
Sometimes if the cumulative method cannot be used directly, it can be reduced and transformed into this form by substitution or other techniques anyway, and then changes the problem to a situation where we can use this method to solve it.

2.2.3 Induction Method

Basically it is what it says, we somehow guess the pattern of the sequence and use induction to prove results we observed. We can combine our guessing with induction as a tool, to either prove a recursive relation, or given a recursive relation and try to verify the statement of the general term formula.
But I want to point out that, sometimes the general term formula is complex and we don't need to waste time trying to figure it out. Sometimes we only need to use the recursive equation. It depends on the situation and you should not just do something because you learned that you could do something and that just came to mind. It might just make the problem more complicated.
It is the same with substitution, reduction or induction. For example when we are reducing the problem, and substituting with a new problem, how do we know that the reduction makes the problem simpler? Even if it's simpler, how do we make sure we can substitute back? Even if the change can be reversed, but because the transformation is not equivalent, so that the new claim we tried to prove might not even be working in the first place. The same holds for induction, many times we try to tackle a problem using the trick, while the problem doesn't even have an inductive structure, and should be treated in direct ways.
Therefore, one has to solve problems on their own and feel what's the best choice, when to use induction and when not to. It takes much experience to learn this.

2.3 Generating Functions 101

2.3.1 Operation Algorithm of Generating Function

We have already defined what generating functions are, so we consider now the four arithmetic operations of generating functions where $n \to \infty$. (If n is just some specific number, then the operations just works as polynomials)
When the form converges, we can just treat it as a normal function to operate, but we generally don't consider the problem of convergence. Thus we have the following regulations:

Addition and subtraction:

$$\sum_{k=0}^{\infty} a_k x^k \pm \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} (a_k \pm b_k) x^k$$

Multiplication:

$$(\sum_{k=0}^{\infty} a_k x^k)(\sum_{k=0}^{\infty} b_k x^k) = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k b_{n-k}) x^n$$

Division: If $f = gh$, then define $\frac{f}{g} = h$

2.3.2 Typical Generating Function Formulas

Example 1 (very useful formula):

$$\frac{1}{1-x} = 1 + x + x^2 + ...$$

Proof: Since

$$(1-x)(1 + x + x^2 + ...) = 1$$

therefore by division law we are done.
Of course, just as we said we crave the form to converge, so it suffices that $|x| < 1$, but this won't cause problems when we are applying the formula to solve actual problems.

Example 2 (very useful formula):

$$\frac{1}{(1-x)^n} = \binom{n-1}{n-1} + \binom{n}{n-1}x + \binom{n+1}{n-1}x^2 + \binom{n+2}{n-1}x^3 + ... = \sum_{k=0}^{\infty}\binom{n-1+k}{n-1}x^k$$

Proof 1: Induction on n
Proof 2: The result can be directly obtained just by computing n-1 derivatives on both sides then dividing by (n-1)!. We won't show the details of the computations, but it's certainly not too complex.

Also note that as a result of example 2, functions of type $\frac{1}{(a-bx)^n}$ can also be extended and evaluated, as it can be transformed and reduced into $\frac{1}{a^n(1-\frac{b}{a}x)^n}$.

2.3.3 Partial Fraction Decomposition

$$\frac{1}{\text{product of first order expressions}}$$

can be Partial Fraction Decomposed into

$$\sum \frac{1}{(\text{first order expression})^n}$$

Of course, we are not limited to 1 in the numerator, since 1 can be replaced by some other function insead.
More details of Partial Fraction Decomposition are well written in advanced high school textbooks, so we will not talk much about it here.

2.3.4 Solving Recursive Equations Using Generating Functions (or vice versa)

One can solve recursive equations using knowledge we just gained about generating functions, by transferring the data of a sequence to the coefficient of a generated function, and then doing manipulations from there.

Example: Given $a_0 = -1, a_1 = 1, a_n = 2a_{n-1} + 3an - 2 + 3^n (n \geq 2)$, find a_n

Solution: Generate the function $f(x)$ of the sequence, i.e

$$f(x) = a_0 + a_1 x + a_2 x^2 + ...$$

Now, leveraging our original condition yields

$$(1 - 2x - 3x^2)f(x) = -1 + 3x + 3^2x^2 + 3^3x^3 + ... = -1 + \frac{3x}{1-3x} = \frac{6x-1}{1-3x}$$

Thus

$$f(x) = \frac{6x-1}{(1+x)(1-3x)^2} = \frac{A}{1+x} + \frac{B}{(1-3x)^2} + \frac{C}{1-3x}$$

Multiplying by 1+x on both sides then let x=-1 yields

$$A = \frac{6x-1}{(1-3x)^2} = -\frac{7}{16}$$

Similarly multiplying by $(1-3x)^2$ on both sides and letting $x = \frac{1}{3}$ yields $B = 3/4$
But this trick won't work for C, since the denumerator for the B term will be zero and such operation is not allowed.
What we do instead is multiplying by x on both sides, then let $x \to \infty$, we have

$$0 = \lim_{x\to\infty} \frac{x(6x-1)}{(1-x)(1-3x)^2} = \lim_{x\to\infty} (\frac{Ax}{1+x} + \frac{Bx}{(1-3x)^2} + \frac{Cx}{1-3x}) = A - \frac{1}{3}C$$

Therefore

$$C = 3A = -\frac{21}{16}$$

Now we substitute A,B,C back to the expression, and leverage the formulas we obtained previously in the basic knowledge part, to get

$$f(x) = -\frac{7}{16(1+x}+\frac{3}{4(1-3x)^2}-\frac{21}{16(1-3x)} = -\frac{7}{16}\sum_{n=0}^{\infty}(-1)^n x^n+\frac{3}{4}\sum_{n=0}^{\infty}\binom{n+1}{1}3^n x^n-\frac{21}{16}\sum_{n=0}^{\infty}3^n x^n =$$

$$= \sum_{n=0}^{\infty} \frac{-7(-1)^n + 4\times 3(n+1)3^n - 21\times 3^n}{16} x^n = \sum_{n=0}^{\infty} \frac{(4n-3)3^{n+1} - 7(-1)^n}{16} x^n$$

Thus

$$a_n = \frac{(4n-3)3^{n+1} - 7(-1)^n}{16}$$

and we are done.

3 Applications of the recursive/generating function

3.1 Fibonacci Sequence

Consider the Fibonacci sequence, defined by

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} (n \geq 2) \tag{12}$$

Here we will present a method for finding a closed formula for the sequence using generating functions. Consider the sequence's generating function

$$F(x) = \sum_{n\geq 0} F_n x^n$$

.
By using (7) one finds that

$$F(x) = x + \sum_{n\geq 2} F_{n-1}x^n + \sum_{n\geq 2} F_{n-2}x^n$$

and by changing the indices

$$F(x) = \sum_{n\geq 0} F_n x^{n+2} + \sum_{n\geq 0} F_n x^{n+1} = x + xF(x) + x^2F(x)$$

.
Thus

$$F(x) = \frac{x}{1-x-x^2}$$

. We will now use partial fractions to rewrite this expression. This decision is quite easily motivated. By doing this we will have a sum of two fractions with linear functions in the denominators, which we already have series expansions for in chapter –! We do this (the details are left as an exercise as space is limited) and find that

$$\frac{x}{1-x-x^2} = \frac{1}{r_1 - r_2}\left(\frac{1}{1-r_1x} - \frac{1}{1-r_2x}\right)$$

where $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$.
Now expanding these fractions we get

$$F(x) = \sum_{n\geq 0} \frac{1}{r_1 - r_2}(r_1^n - r_2^n)x^n$$

Hence we find that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{\sqrt{5}-1}{2}\right)^n\right)$$

which is the sought closed formula.

3.2 Combinatorics: counting problem

Counting problems in combinatorics can be solved using many methods, but here we constrain ourselves to recursive equations, and generating function only.
We present the application of recursive equations in counting problems first.
When the desired quantity of counting is related to natural numbers, then the quantity can be carried in the form of a sequence, and then the general term formula of the sequence can be obtained by establishing recursive equations, and as we solve the equations, the original counting problem will then be answered.
When we use the recursive equation to solve a counting problem, there are usually three steps:

1) Find the initial value
2) Find the recurrence relation
3) Solve the recurrence equation

1) is usually simple, since it often only involves enumeration of small numbers, which just needs some simple case discussion.
But of course, careless mistakes in 1) have to be taken care of when case bashing, one should verify that it is indeed no repetition and no omission. To do this, a person can for example follow a logically correct principle when classifying and discussing the small cases.
For 3), we have already developed useful tools in the basic knowledge part. But of course, sometimes we encounter non routine problems, and need to use creativity and do algebraic manipulations on our own, in order to solve the equation.
The hardest part often lies in 2), i.e which quantity we should define a sequence on to be included in the equations, since in many cases, we cannot directly define the required quantity, and sometimes, it is not enough for us to define just one quantity, and we often need to continue to add more auxiliary variables.
However, after we define the variables, it is not so difficult to analyze what equation relationship they need to satisfy. Basically, it is often through constructing some combinatorial model to assume various situations, and then add up the numbers of each situation.
Now, let us now try to leverage recursive equations to tackle a counting problem.

Example 1: A circle is divided into $n \geq 2$ sectors $S_1, S_2, ..., S_n$ and we have $m \geq 2$ colors, each sector should be dyed in exactly one color, find the number of all coloring schemes such that adjacent sectors are dyed with different colors.

Solution: Fix m and denote a_n by the number of such coloring in the setting of n circle sectors.

1) Find the initial value.
When $n = 2$, we have

$$a_2 = m(m-1)$$

2) Find the recurrence relation.
By additive principle we have

$$m(m-1)^{n-1} = a_n + a_{n-1}$$

This is due to that the number $m(m-1)^{n-1}$ can seen as S_1 has m choice of coloring, S_2 have m-1 choice, S_3 also have m-1 choice, etc... until S_n also have m-1 choice. This coloring scheme secures that all neighbors except S_1 and S_n have different colors, and indeed the number covers all such coloring I just described.
Thus the number can be split into two cases, where it is discussed whether S_1 and S_n have the same color or not. If they don't have the same color, then number of such cases is a_n. Otherwise we can merge S_1 and S_n and notice that now the number of such coloring is a_{n-1} since each

adjacent sector has different colors. Piecing the two cases together yields the equation.

3) Solve the recurrence equation.
Substitute and let

$$b_n = \frac{a_n}{(m-1)^n}$$

now

$$b_n = -\frac{1}{m-1}b_{n-1} + \frac{m}{m-1}$$

Notice that it has structure of

$$x_n = px_{n-1} + q$$

We have mentioned earlier that this is a first-order linear (inhomogeneous) recursive sequence, and in the basic knowledge section we have already solved it. Therefore the rest is trivial, and by substitution method we mentioned in the knowledge section we can reverse engineer a_n from the solution of b_n. Thus we are done.

Now, let us move on to the application of generating functions in counting problems.
Generating functions is very useful to model various combinatorial settings where we want to count something.
We often transform the data that needs to be counted into a sequence that again can be generated into a function by matching the sequence as coefficient to a power series, where we can then leverage the form and knowledge we have about generating functions to find the final answer.

Exemple 2: Find the number of three-digit positive integers whose sum of the three digits is equal to 17.

Solution: The problem can be translated into finding number of integral a, b, c so that $a + b + c = 17$ where $1 \leq a \leq 9, 0 \leq b, c \leq 9$,

We can again transfer the carrier of the data into a function, so that the data now interprets as some element of that function, in our case the element stands for the coefficients in:

$$(x^1 + x^2 + ... + x^9)(1 + x + ... + x^9)(1 + x + ... + x^9)$$

Thus our goal is now to find the coefficient of x^{17} of the expression above, since it represents the number of combinations that sums up to be 17 with our constrain that $1 \leq a \leq 9, 0 \leq b, c \leq 9$, manifested as the exponent of terms in the brackets.
Notice that we have now transformed a combinatorial counting problem into some specific algebraic problem!

We want to continue by writing the brackets into a somewhat more closed form. We can just use the results we have for summation of a geometric sequence, because it's just as we said in the basic knowledge part, when the terms are limited, we can just treat it as normal polynomials.
Thus we have

$$\frac{x(1-x^9)(1-x^{10})^2}{(1-x)^3} \tag{13}$$

and want to find out the coefficient of x^{17} from it.

We can now continue the solution in two ways, either through treating it as it is in closed form with polynomial division, and using the algorithm to find out find out the coefficient of x^{17}, or we can interpret the denumerator as a formal power series and leverage the tricks we have learned about generating functions and expand the closed form. This time the open form will be switched into a form where it is easier to extract the data we want to know, in our case the coefficient. This shows the power of generating functions.
Of course, we choose the second route. Remember the useful formula we proved at the basic knowledge part? We have:

$$\frac{1}{(1-x)^n} = \binom{n-1}{n-1} + \binom{n}{n-1}x + \binom{n+1}{n-1}x^2 + \binom{n+2}{n-1}x^3 + ... = \sum_{k=0}^{\infty}\binom{n-1+k}{n-1}x^k$$

and take n=3 in particular.

So that Equation 13 becomes

$$(x - x^{10} - 2x^{11} + 2x^{20} + x^{21} - x^{30})\sum_{k=0}^{\infty}\binom{3-1+k}{3-1}x^k$$

after expanding the numerator, and using our formula on denumerator.

Therefore the coefficient of x^{17} is

$$\binom{18}{2} - \binom{9}{2} - 2\binom{8}{2} = 153 - 36 - 56 = 61$$

This completed the proof and we are done.

4 Final Notes

There are other applications where the generating functions could be in use. One of the exemples is that we often use generating functions to prove combinatorial identities. For instance, Vandermonde's identity is a good example.

On the other hand, recursion is more of a problem-solving idea rather than a tool. The method of solving recursive equations we introduce in this article is more of a technique of a topic named sequence in algebra rather than recursion.

The idea of recursion itself should be the most used in programming, but it is also reflected in mathematics competitions in many places, not just in combination counting problems. For example, for two-player games, we often use recursion to solve (positional analysis, i.e using recursion to engineer the winning and losing positions). Even the functions of the integer in functional equations, where we can consider recursive chains, and then use this idea to tackle the problems.