Karamata's inequality and its applications

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1 Introduction

In mathematics, we often encounter expressions that vary or are unwieldy. For these, inequalities are important tools for consolidating dependencies and reducing complexity while preserving key information. This document aims to explore elementary inequalities resulting from the particulars of the shape of convex functions. The goal is to educate the reader in the application of Karamata's inequality and its special case Jensen's inequality in Olympiad algebra.

1.1 Convex functions

To see the defining property of convex functions we observe the graph of a function. A function is said to be convex if for any two points on its graph, the line segment connecting those points is above the graph of the function. For example, the function $f(x) = x^2 + 1$ is convex, while the function $g(x) = x^3 - x$ is not.

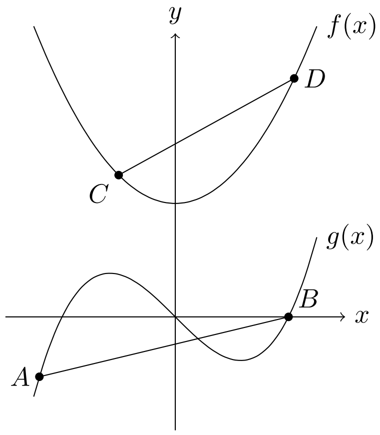

Figure 1: The function $f(x) = x^2 + 1$ is convex, but the function $g(x) = x^3 - x$ is not as the segment between A and B does not lie above the graph of the function.

This is often formulated as an inequality between the y-coordinates of the segment and of the graph $y = f(x)$ by parameterizing all points on the segment.

Definition 1.1. A function $f : \mathbb{R} \to \mathbb{R}$ is convex on an interval I if

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b) \quad \forall a, b \in I, a \neq b, \lambda \in (0,1)\,.$$

Replacing the $\leq$ sign with $<$ we obtain the definition of a strictly convex function. Concave functions f are functions for which $-f$ is convex, which in practice can be thought of as convex functions with all inequalities reversed. Convex functions over the real numbers are necessarily continuous, but not necessarily differentiable.

1.2 Proving convexity

Proving convexity for some functions such as $e^{\frac{1}{\sqrt{x}}}$ from the inequality definition is difficult. We therefore present some alternative tools for proving convexity.

Lemma 1.1. *A function f is convex if and only if the symmetric function $R(x, y) = \frac{f(x)-f(y)}{x-y}$ is non-decreasing in x for every fixed y.*

Lemma 1.2. *A differentiable function f is convex if and only if f' is non-decreasing. If the function is twice differentiable, this is in turn equivalent to $f'' \geq 0$.*

Lemma 1.3. *If the functions g and h are convex and g is non-decreasing, the function $f = g \circ h$ is also convex.*

When proving convexity for a function in Olympiad problems, using Lemma 1.2 is the most common. In the following example, we illustrate the use of all three methods.

Example 1. Show that the functions $\frac{1}{\sqrt{x}}$, e^x, and $e^{\frac{1}{\sqrt{x}}}$ are convex for $x > 0$.

Solution. Fix y and consider

$$R(x,y) = \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}}}{x-y} = \frac{\sqrt{y}-\sqrt{x}}{\sqrt{xy}} \frac{1}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = \frac{-1}{\sqrt{xy}(\sqrt{x}+\sqrt{y})}$$

which is increasing in x as $\sqrt{xy}(\sqrt{x}+\sqrt{y})$ is increasing in x. Hence $\frac{1}{\sqrt{x}}$ is convex for $x > 0$ by Lemma 1.1. Since the second derivative of e^x is e^x which is positive for all real x, the function e^x is convex for all real x by Lemma 1.2. Finally, since $g(x) = e^x$ is convex and non-decreasing and $h(x) = \frac{1}{\sqrt{x}}$ is convex, we get by Lemma 1.3 that $f(x) = g \circ h = e^{\frac{1}{\sqrt{x}}}$ is convex. □

Exercise 1. Prove that $|x|$ is convex.

Exercise 2. Prove that $\ln x$ is concave.

Exercise 3. Prove that $\sin(\cos x)$ is concave on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Exercise 4. Prove that $e^{x^4-x^3+x^2-x+1}$ is convex.

1.3 Majorization

To state Karamata's inequality, we need to introduce the concept of majorization for two sequences of numbers. Majorization is a part of not only Karamata's inequality, but also other inequalities such as Muirhead's inequality.

Definition 1.2. Given two sequences $a_1, \dots, a_n$ and $b_1, \dots, b_n$ such that $a_1 \geq \dots \geq a_n$ and $b_1 \geq \dots \geq b_n$, we say that the sequence (a_i) weakly majorizes the sequence (b_i) if for every $1 \leq k \leq n$, $\sum_{i=1}^k a_i \geq \sum_{i=1}^k b_i$. If additionally we have $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, then we say that (a_i) majorizes (b_i) and write $(a_i) \succ (b_i)$.

Example 2. The sequence $(3, 2, 1)$ majorizes the sequence $(2, 2, 2)$ as $3 \geq 2$, $3 + 2 \geq 2 + 2$ and $3 + 2 + 1 = 2 + 2 + 2$.

Exercise 5. Given two sequences $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ such that $(a_i) \succ (b_i)$, what can be said about a_1 and b_1? a_n and b_n? a_i and b_i for $2 \leq i \leq n-1$?

Exercise 6. If f and g are two functions defined on an interval I such that $f(x) \geq g(x)$ for every $x \in I$ and $a_1, \dots, a_n \in I$, prove that the sequence $(f(a_i))$ weakly majorizes $(g(a_i))$.

1.4 Karamata's inequality

With the concepts of convex functions and majorization introduced, we can state and prove Karamata's inequality.

Theorem 1.1 (Karamata's inequality)**.** *Let f be a function that is convex on some interval I of the real line. If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are numbers in I such that $(a_i) \succ (b_i)$, then*

$$f(a_1) + \dots + f(a_n) \geq f(b_1) + \dots + f(b_n).$$

Proof. From the majorization we have that $a_1 \geq a_2 \geq \dots \geq a_n$ and $b_1 \geq b_2 \geq \dots \geq b_n$. Let $c_i = \frac{f(a_i)-f(b_i)}{a_i-b_i}$ for $i = 1, \dots, n$. From Lemma 1.1, the quantity $R(x,y) = \frac{f(x)-f(y)}{x-y}$ is increasing in x and in y, hence

$$c_{i+1} = R(a_{i+1}, b_{i+1}) \leq R(a_i, b_{i+1}) \leq R(a_i, b_i) = c_i$$

Further let $A_0 = B_0 = 0$ and $A_i = a_1 + \cdots + a_i$ and $B_i = b_1 + \cdots + b_i$ for $i = 1, \ldots, n$, then since (a_i) majorizes (b_i), $A_i \geq B_i$ for every i. Now note

$$\sum_{i=1}^{n} f(a_i) - f(b_i) = \sum_{i=1}^{n} \frac{f(a_i) - f(b_i)}{a_i - b_i}(a_i - b_i) = \sum_{i=1}^{n} c_i(a_i - b_i) = \sum_{i=1}^{n} c_i(A_i - A_{i-1} - B_i + B_{i-1})$$

Breaking up the summation and using $A_0 = B_0$ and $A_n = B_n$

$$\begin{aligned}\sum_{i=1}^{n} f(a_i) - f(b_i) &= \sum_{i=1}^{n} c_i(A_i - B_i) - \sum_{i=1}^{n} c_i(A_{i-1} - B_{i-1}) \\ &= c_n(A_n - B_n) + \sum_{i=1}^{n-1}(c_i - c_{i+1})(A_i - B_i) - c_1(A_0 - B_0) \\ &= \sum_{i=1}^{n-1}(c_i - c_{i+1})(A_i - B_i) \geq 0 \qquad (1)\end{aligned}$$

Where we used that $c_i \geq c_{i+1}$ and $A_i \geq B_i$ in the last step. Hence we conclude that $\sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} f(b_i)$. □

Looking at the sum in (1), equality in Karamata's inequality is obtained if, for every i, $A_i = B_i$ or $c_i = c_{i+1}$, the latter of which is equivalent to f being linear on $[\min(a_{i+1}, b_{i+1}), \max(a_i, b_i)]$. Note that if f is strictly convex, it is impossible for f to be linear on some interval, so in that case, equality occurs if $A_i = B_i$ for every i, which is equivalent to $a_i = b_i$ for every i.

Given $a_1 \geq \cdots \geq a_n$ with arithmetic mean $m = \frac{a_1+\cdots+a_n}{n}$, the sequence $(a_1, \ldots, a_n)$ majorizes the sequence $(m, \ldots, m)$. This yields a common special case of Karamata's inequality called Jensen's inequality.

Theorem 1.2 (Jensen's inequality)**.** *For a function f that is convex on an interval I of the real line, the inequality*

$$\frac{f(a_1) + \cdots + f(a_n)}{n} \geq f\left(\frac{a_1 + \cdots + a_n}{n}\right)$$

holds for all $a_1, \ldots, a_n \in I$.

Jensen's inequality has a generalization in which weights are assigned to every term. We present the following geometric proof, although proofs using induction or Theorem 1.4 are possible.

Theorem 1.3 (Weighted Jensen's inequality)**.** *Let f be a function convex in an interval I. Given weights $w_1, w_2, \ldots, w_n \in (0, 1)$ such that $\sum_{i=1}^{n} w_i = 1$, the inequality*

$$w_1 f(a_1) + w_2 f(a_2) + \cdots + w_n f(a_n) \geq f(w_1 a_1 + w_2 a_2 + \cdots + w_n a_n)$$

holds for all $a_1, a_2, \ldots, a_n \in I$.

Proof. We rephrase the inequality as the following lemma.

Lemma 1.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and let S be a set of points of the form $(x, f(x))$. For any point (p, q) inside the convex hull of S we have that $f(p) \leq q$.*

Proof. As (p, q) is inside the convex hull of S, there must exist 3 points in S that form a triangle containing (p, q). Consider the side of this triangle directly below (p, q), which is a segment between two points $(a, f(a), (b, f(b)) \in S$ with $a \leq p \leq b$. By the geometric definition of a convex function, when $x \in [a, b]$, the graph $y = f(x)$ is below the segment between points $(a, f(a))$ and $(b, f(b))$, which is below the point (p, q). Hence (p, q) is above the graph $y = f(x)$, which means that $f(p) \geq q$. □

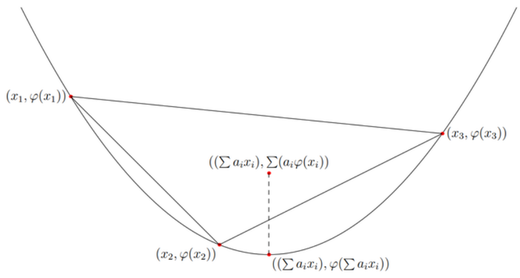

Figure 2: A proof without words of Jensen's inequality by LazarIlic0.

Noting that any center of mass of the points $P = \{(x, f(x)) \mid x \in \{a_1, \ldots, a_n\}\}$ must lie inside their convex hull, we give every point $(a_i, f(a_i))$ a point mass of w_i. By Lemma 1.4 we have that $f(w_1 a_1 + w_2 a_2 + \ldots + w_n a_n) \leq w_1 f(a_1) + w_2 f(a_2) + \ldots + w_n f(a_n)$. □

Similarly, Karamata's inequality also has a weighted version due L. Fuchs, although this inequality rarely comes up in Olympiad algebra.

Theorem 1.4 (Weighted Karamata's inequality)**.** *Let f be function convex on an interval I. Let $w_1, \ldots, w_n$ be real numbers and let $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$ be two sequences with numbers belonging to I such that $(a_i w_i) \succ (b_i w_i)$. Then*

$$w_1 f(a_1) + \cdots + w_n f(a_n) \geq w_1 f(b_1) + \cdots + w_n f(b_n)$$

Proof. The proof is mostly the same as the proof of the unweighted variant. Let $c_i = \frac{f(a_i) - f(b_i)}{a_i - b_i}$, $A_i = w_1 a_1 + \cdots + w_i a_i$ and $B_i = w_1 b_1 + \cdots + w_i b_i$. Let $A_0 = B_0 = 0$ and notice that

$$\sum_{i=1}^{n} w_i f(a_i) - w_i f(b_i) = \sum_{i=1}^{n} c_i (w_i a_i - w_i b_i) = \sum_{i=1}^{n} c_i (A_i - A_{i-1} - B_i + B_{i-1})$$

which is non-negative as $c_{i+1} \leq c_i$ and $A_i \geq B_i$. The conclusion follows with the same equality cases as the unweighted variant. □

Exercise 7. Given real numbers $a_1 \geq \cdots \geq a_n$ with arithmetic mean $m = \frac{a_1 + \cdots + a_n}{n}$, verify that $(a_1, \ldots, a_n) \succ (m, \ldots, m)$ and conclude Jensen's inequality from Karamata's inequality.

2 Applying Karamata's inequality

Two things are needed to apply Karamata's inequality: a convex function, and two sequences, one majorizing the other. Here, we present common ideas used for finding these two things and proving inequalities with Karamata.

Example 3. In an acute triangle with angles α, β, γ, prove that

$$2 \leq \sin \alpha + \sin \beta + \sin \gamma \leq \frac{3\sqrt{3}}{2}$$

Proof. WLOG assume that $\frac{\pi}{2} > \alpha \geq \beta \geq \gamma > 0$ as the triangle is acute. Note that $\alpha + \beta + \gamma = \pi$ is constant, hence $f(x) = \sin x$ is most likely a good choice of a function. As $f''(x) = -\sin x \leq 0$ we have that f is concave. From $(\frac{\pi}{2}, \frac{\pi}{2}, 0) \succ (\alpha, \beta, \gamma) \succ (\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ we obtain the desired inequalities,

$$\sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin 0 = 2 \leq \sin\alpha + \sin\beta + \sin\gamma \leq \frac{3\sqrt{3}}{2} = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{\pi}{3}$$

keeping in mind that the inequalities are flipped due to the concavity of f. □

Finding a set of variables whose sum is constant is often the first step towards a solution using Karamata's or Jensen's inequality.

Exercise 8. In an acute triangle with angles α, β, γ, prove that

$$1 \leq \cos\alpha + \cos\beta + \cos\gamma \leq \frac{3}{2}$$

Exercise 9 (Modified version of SMT Final 2017)**.** Let a, b, c be side-lengths in an arbitrary triangle with area A. Find the largest constant K such that

$$K \leq \frac{ab + bc + ca}{A}$$

2.1 Using logarithms

Assume that we wish to prove an inequality that can be written in the form

$$f(a_1)f(a_2)\cdots f(a_n) \geq f(b_1)f(b_2)\cdots f(b_n). \tag{2}$$

We cannot use Karamata directly on f and the sequences (a_i), (b_i) since the values are being multiplied instead of summed. However, using some properties of the logarithm, the inequality can be transformed into a form where Karamata or Jensen can be used. Recall the following properties for $x, y > 0$:

1. $x \geq y \iff \ln x \geq \ln y$.
2. $\ln x \cdot y = \ln x + \ln y$.
3. For any $a \in \mathbb{R}$, $\ln x^a = a \cdot \ln x$.
4. The derivative of $\ln x$ is $\frac{1}{x}$.

The first three properties hold for logarithms with any base, but the last property makes using the natural logarithm the most convenient. Going back to (2), it is equivalent to showing

$$\ln(f(a_1)f(a_2)\cdots f(a_n)) \geq \ln(f(b_1)f(b_2)\cdots f(b_n))$$

by the first property. By the second property, we can write this inequality as

$$\ln f(a_1) + \ln f(a_2) + \cdots + \ln f(a_n) \geq \ln f(b_1) + \ln f(b_2) + \cdots + \ln f(b_n)$$

and letting $g(x) = \ln f(x)$, the inequality we wish to prove writes

$$g(a_1) + g(a_2) + \cdots + g(a_n) \geq g(b_1) + g(b_2) + \cdots + g(b_n).$$

The inequality is now of the correct form for using Karamata, and verifying the conditions required for using the theorem, the inequality follows. We illustrate this technique further with a concrete example.

Example 4 (Tip_pay on AOPS[1], modified)**.** The sum of the positive numbers $x_1, x_2, \ldots, x_n$ is equal to $\frac{1}{2}$. Prove that

$$\frac{1 - x_1}{1 + x_1} \cdot \frac{1 - x_2}{1 + x_2} \cdots \frac{1 - x_n}{1 + x_n} \geq \left(\frac{2n - 1}{2n + 1}\right)^n.$$

[1] https://artofproblemsolving.com/community/c6h2722951p23691396

Solution. The function $\ln \frac{1-x}{1+x}$ has second derivative $-\frac{4x}{(x^2-1)^2} < 0$ for $x > 0$, so it is concave. Thus, by Jensen we have

$$\ln \frac{1-x_1}{1+x_1} + \ln \frac{1-x_2}{1+x_2} + \cdots + \ln \frac{1-x_n}{1+x_n} \geq n \ln \frac{1-\frac{x_1+\cdots+x_n}{n}}{1+\frac{x_1+\cdots+x_n}{n}} = n \ln \frac{1-\frac{1}{2n}}{1+\frac{1}{2n}} = \ln \left(\left(\frac{2n-1}{2n+1} \right)^n \right)$$

so exponentiating both sides yields the desired inequality.[2] □

Defining new variables after applying logarithms can also be of use for finding a sequence that majorizes another, as seen in the following example.

Example 5 (IMOmath)**.** Let $a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq \ldots \geq b_n$ be sequences of positive real numbers such that $a_1 \cdots a_k \geq b_1 \cdots b_k$ for $k = 1, \ldots, n-1$ and $a_1 \cdots a_n = b_1 \cdots b_n$. Prove that

$$a_1 + a_2 + \ldots + a_n \geq b_1 + b_2 + \ldots + b_n.$$

Solution. Perform the change of variables $x_i = \ln a_i$ and $y_i = \ln b_i$. Taking logarithms on both sides of the given inequalities gives

$$x_1 + \ldots + x_k \geq y_1 + \ldots + y_k$$

for all k with equality for $k = n$, so $(x_i) \succ (y_i)$. Now the inequality we wish to prove is

$$e^{x_1} + e^{x_2} + \cdots + e^{x_n} \geq e^{y_1} + e^{y_2} + \cdots + e^{y_n}$$

but this follows from Karamata with $f(x) = e^x$, a convex function. □

Exercise 10 (AM-GM-inequality)**.** Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

2.2 Proving majorization

Sometimes, finding the two sequences such that one majorizes the other is not obvious or may require additional work. The terms of the sequences may include more than one of the variables, as long as one can prove majorization. We illustrate some key ideas in the following examples.

Example 6 (APMO 1996)**.** Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$$

for a, b, c being the sides of a triangle.

Solution. The inequality is symmetric in all variables, so assume without loss of generality that $a \geq b \geq c$. The function $f(x) = \sqrt{x}$ is concave for $x \geq 0$, so the inequality is proven by Karamata if we can show that $(a+b-c, c+a-b, b+c-a) \succ (a, b, c)$. Note first that since $a \geq b \geq c$, we have

$$a + b - c \geq c + a - b \geq b + c - a$$

so we know the orderings of the two sequences. All that remains for majorization is proving:

$$\begin{aligned} a + b - c &\geq a \\ (a + b - c) + (c + a - b) &\geq a + b \\ (b + c - a) + (c + a - b) + (a + b - c) &= a + b + c \end{aligned}$$

but these inequalities are clear. □

[2]The left hand side of the original problem was just $\frac{1}{3}$, but in our case it is possible to prove $f(n) = (\frac{2n-1}{2n+1})^n \geq \frac{1}{3}$ for all $n \geq 1$ by showing that f is increasing. In fact we have the limit $\lim_{n\to\infty} f(n) = \frac{1}{e} \approx 0.368$.

The next example has a similar structure, but requires slightly more work.

Example 7 (Ngankaka on AOPS[3])**.** Given $a, b, c > 0$, prove that

$$2\sqrt{a^2+b^2+c^2} + \sqrt{ab+bc+ac} \geq \sqrt{a^2+ab+b^2} + \sqrt{b^2+bc+c^2} + \sqrt{c^2+ca+a^2}.$$

Solution. Again the inequality is symmetric, so we may assume that $a \geq b \geq c$. The inequality can be written as

$$f(a^2+b^2+c^2)+f(a^2+b^2+c^2)+f(ab+bc+ac) \geq f(a^2+ab+b^2)+f(b^2+bc+c^2)+f(c^2+ca+a^2)$$

where $f(x) = \sqrt{x}$, a concave function for $x > 0$, so if we can prove that,

$$(a^2+ab+b^2, b^2+bc+c^2, c^2+ca+a^2) \succ (a^2+b^2+c^2, a^2+b^2+c^2, ab+bc+ca)$$

we are done by Karamata. Since we assumed $a \geq b \geq c$, the first sequence is ordered as

$$a^2+ab+b^2 \geq c^2+ca+a^2 \geq b^2+bc+c^2.$$

For the second sequence notice that $(a^2+b^2)/2 \geq ab$ by AM-GM, which gives the ordering

$$a^2+b^2+c^2 \geq a^2+b^2+c^2 \geq ab+bc+ca.$$

This means that we have to prove:

$$\begin{aligned} a^2+ab+b^2 &\geq a^2+b^2+c^2 \\ (a^2+ab+b^2)+(c^2+ca+a^2) &\geq 2(a^2+b^2+c^2) \\ (a^2+ab+b^2)+(c^2+ca+a^2)+(b^2+bc+c^2) &= 2(a^2+b^2+c^2)+ab+bc+ca \end{aligned}$$

but the two inequalities follow from $ab \geq b^2 \geq c^2$ and $ca \geq c^2$, and the last equality is clear. □

Exercise 11. Prove for $a, b, c > 0$ the inequality

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \leq \frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}.$$

Exercise 12. Prove that the inequality

$$\cos(2x_1-x_2)+\cos(2x_2-x_3)+\cdots+\cos(2x_n-x_1) \leq \cos(x_1)+\cos(x_2)+\cdots+\cos(x_n)$$

holds for arbitrary numbers x_1, x_2, ... , x_n in the interval $[-\pi/6, \pi/6]$.

2.3 Finding a function

Further algebraic manipulation is often needed in order to rewrite expressions into forms to which we can apply Karamata. To that end we present some key ideas including; introducing dependence on a single quantity, homogeneity and simplifying substitutions. In Olympiad algebra, homogeneity is when all expressions have equal degree so that a scaling of all variables by a constant yields an equivalent equation.

Example 8 (IMO 2009 Shortlist A2)**.** Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = a+b+c$. Prove that

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(2b+c+a)^2}+\frac{1}{(2c+a+b)^2} \leq \frac{3}{16}.$$

[3]https://artofproblemsolving.com/community/c6h2629634p22734355

Solution. There is no obvious function f and pair of sequences that we can choose here, but by manipulating the inequality, this will appear. First we homogenize the inequality by multiplying the the right hand side by $(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})/(a+b+c) = 1$ to obtain

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(2b+c+a)^2}+\frac{1}{(2c+a+b)^2} \leq \frac{3(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})}{16(a+b+c)}.$$

As the inequality is homogeneous, a scaling of all variables does not change the value of either side. We may therefore impose a new condition on the variables, so let $a+b+c=3$. The inequality becomes

$$\frac{1}{(a+3)^2}+\frac{1}{(b+3)^2}+\frac{1}{(c+3)^2} \leq \frac{(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})}{16}$$

which can now be written as

$$f(a)+f(b)+f(c) \geq 0 = 3 \cdot f(1) = 3 \cdot f\left(\frac{a+b+c}{3}\right)$$

where $f(x) = \frac{1}{16x} - \frac{1}{(x+3)^2}$, and this is exactly the form of Jensen's inequality. All that remains is checking that this function is convex on the interval $(0,3)$. We have $f''(x) = \frac{1}{8x^3} - \frac{6}{(x+3)^4}$ which is decreasing on $(0,3)$, and $f''(3) = 0$ so f is convex and the desired inequality follows. □

Example 9. If a,b,c are positive real numbers, prove that

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}.$$

Proof. Introduce the quantity $S = a+b+c$ and rewrite all summands to be univariate.

$$\frac{a}{(S-a)^2}+\frac{b}{(S-b)^2}+\frac{c}{(S-c)^2} \geq \frac{9}{4S}$$

Now consider $f(x) = \frac{x}{(S-x)^2}$ and notice that $f''(x) = \frac{4S+2x}{(S-x)^4} > 0$ as well as $f(S/3) = \frac{S/3}{(S-S/3)^2} = \frac{9S}{3 \cdot 4S^2} = \frac{3}{4S}$. The inequality becomes $f(a)+f(b)+f(c) \geq 3f(S/3)$ which follows by Jensen. □

Exercise 13 (Nesbitt's inequality)**.** Prove for positive real numbers a,b,c the inequality

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \geq \frac{3}{2}.$$

Exercise 14 (IMOmath[4])**.** If x,y,z are positive real numbers such that $x+y+z = xyz$ prove that

$$\frac{1}{1+xy}+\frac{1}{1+yz}+\frac{1}{1+zx} \leq \frac{3}{4}.$$

3 Further example problems

This section contains further examples of the techniques presented as well as some smaller tricks.

Example 10 (Korrespondenskurs 22/23)**.** Let $x_0, x_1, \ldots, x_n$ be positive real numbers such that $x_0 > x_1 > x_2 > \cdots > x_n$. Prove that

$$\frac{(x_0-x_1)(x_1-x_2)\cdots(x_{n-1}-x_n)}{(x_0+x_1)(x_1+x_2)\cdots(x_{n-1}+x_n)} \leq \left(\frac{\sqrt[n]{x_0}-\sqrt[n]{x_n}}{\sqrt[n]{x_0}+\sqrt[n]{x_n}}\right)^n.$$

[4]Problem author Zuming Feng

Solution. As $\ln x$ is an increasing function, applying logarithms to both sides reduces the problem to showing

$$\sum_{i=1}^{n} \ln \frac{x_{i-1} - x_i}{x_{i-1} + x_i} \le n \ln \frac{\sqrt[n]{x_0} - \sqrt[n]{x_n}}{\sqrt[n]{x_0} + \sqrt[n]{x_n}}.$$

Further let $a_i = \ln \frac{x_i}{x_{i-1}}$, letting us rewrite the left hand summands in a single variable

$$\sum_{i=1}^{n} \ln \frac{1 - e^{a_i}}{1 + e^{a_i}} \le n \ln \frac{1 - \exp \frac{1}{n} \sum_{i=i}^{n} a_i}{1 + \exp \frac{1}{n} \sum_{i=i}^{n} a_i}$$

noting that $\ln \frac{\sqrt[n]{x_n}}{\sqrt[n]{x_0}} = \frac{1}{n} \sum_{i=i}^{n} a_i$ for the right hand side. The inequality is now on the form of Jensen's inequality. All that remains is to prove the concavity of $f(x) = \ln \frac{1-e^x}{1+e^x}$,

$$f''(x) = \frac{-e^x}{(1+e^x)^2} + \frac{-e^x}{(1-e^x)^2} < 0$$

defined for $x < 0$ while verifying that $a_i < \ln 1 = 0$ as $x_i < x_{i-1}$. □

A powerful observation for Olympiad use is that if a continuous function is not convex, then there exists some interval on which it is concave. On this interval we have that Jensen's inequality does not hold. Hence we could be sure that the function f in Example 10 would be concave and that we had a complete proof before verifying its concavity, since we had reduced the inequality to the form of Jensen's inequality.

Example 11 (Weighted AM-GM)**.** Let $x_1, x_2, \ldots, x_n$ and $w_1, w_2, \ldots, w_n$ be non-negative real numbers such that $\sum_{i=1}^{n} w_i = 1$. Prove that

$$w_1 x_1 + w_2 x_2 + \ldots + w_n x_n \ge x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}$$

Solution. If $x_i = 0$ and $w_i > 0$, then the right hand side is non-negative and the left hand side is 0. If $x_i = 0 = w_i$, then removing both yields an equivalent inequality. Hence we can assume that $x_i > 0$ for all i. Introduce the substitution $a_i = \ln x_i$ and rewrite the inequality as

$$w_1 e^{a_1} + w_2 e^{a_2} + \ldots + w_n e^{a_n} \ge \exp\left(w_1 a_1 + w_2 a_2 + \ldots + w_n a_n\right)$$

which follows by weighted Jensen on $f(x) = e^x$, which is convex as $f''(x) = e^x > 0$. □

Example 12 (Popoviciu's inequality)**.** Let f be a function convex on an interval $I \subseteq \mathbb{R}$ and let $x, y, z \in I$ be three real numbers. Show that

$$\frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \ge \frac{2}{3}\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right].$$

Solution. The inequality is symmetric in all variables, so without loss of generality $x \ge y \ge z$, which implies the ordering $\frac{x+y}{2} \ge \frac{z+x}{2} \ge \frac{y+z}{2}$. As flipping a function over the y-axis preserves convexity we can assume that $y \ge \frac{x+y+z}{3}$. Multiplying the inequality by 3 yields that the inequality is implied by Karamata if

$$\left(x, y, \frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}, z\right) \succ \left(\frac{x+y}{2}, \frac{x+y}{2}, \frac{x+z}{2}, \frac{x+z}{2}, \frac{y+z}{2}, \frac{y+z}{2}\right)$$

As $y \ge \frac{x+y+z}{3}$ we have that $2y \ge x + z$. The majorization is true as $x \ge \frac{x+y}{2}$, $x + y \ge x + y$, $\frac{x+y+z}{3} \ge \frac{x+(x+z)/2+z}{3} = \frac{x+z}{2} \ge \frac{x+y}{2}$ and $x + y + z + 3\frac{x+y+z}{3} = 2(x + y + z)$. □

Notice how we utilized the symmetries of the question to reduce the number of possible cases for majorization down to a single case.

4 Problems

Problem 1. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and let $p \geq 1$. Prove that

$$(a_1 + a_2 + \cdots + a_n)^p \leq n^{p-1}(a_1^p + a_2^p + \cdots + a_n^p).$$

Problem 2. Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\prod_{i=1}^n x_i = 1$. Prove that

$$x_1 + x_2 + \ldots + x_n \geq \sqrt{x_1} + \sqrt{x_2} + \ldots + \sqrt{x_n}.$$

Problem 3 (Korrespondenskurs 04/05)**.** The numbers $a_1, a_2, \ldots, a_n$ are positive and such that $a_1 + a_2 + \cdots + a_n = 1$. Show that

$$\frac{a_1}{1 + a_1\sqrt{2}} + \frac{a_2}{1 + a_2\sqrt{2}} + \cdots + \frac{a_n}{1 + a_n\sqrt{2}} \leq \frac{n}{n + \sqrt{2}}.$$

Problem 4 (AM-HM inequality)**.** Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}.$$

Problem 5. Let a, b, c be positive real numbers. Prove that:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq \frac{9}{a+b+c}.$$

Problem 6 (MOP 04)**.** Show that for all positive reals a, b, c

$$\left(\frac{a+2b}{a+2c}\right)^3 + \left(\frac{b+2c}{b+2a}\right)^3 + \left(\frac{c+2a}{c+2b}\right)^3 \geq 3.$$

Problem 7 (IMOmath)**.** Let $a_1, \ldots, a_n$ be positive real numbers. Prove that

$$(1 + a_1)(1 + a_2)\cdots(1 + a_n) \leq \left(1 + \frac{a_1^2}{a_2}\right) \cdot \left(1 + \frac{a_2^2}{a_3}\right) \cdots \left(1 + \frac{a_n^2}{a_1}\right).$$

Problem 8 (IMO 1999/2)**.** Let n be a fixed integer, with $n \geq 2$. Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_{1 \leq i \leq n} x_i\right)^4$$

holds for all reals numbers $x_1, \ldots, x_n \geq 0$.

Problem 9 (IMO 2000/2)**.** Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$\left(x - 1 + \frac{1}{y}\right)\left(y - 1 + \frac{1}{z}\right)\left(z - 1 + \frac{1}{x}\right) \leq 1.$$

5 Hints

5.1 Hints to exercises

1. Consider the graph and the segment creating a triangle.
2. Consider the derivative of $-\ln x$.
3. Consider the second derivative.
4. Show that the polynomial in the exponent is a convex function.
5. $a_1 \geq b_1$ and $a_n \leq b_n$ are the only things that can be said.
6. Induct on k for $f(a_1) + \ldots + f(a_k) \geq g(a_1) + \ldots + g(a_k)$
7. Show that $\frac{a_1+\ldots+a_k}{k} \geq \frac{a_{k+1}+\ldots+a_n}{n-k}$ for all k.
8. $\cos x$ is concave on $[0, \frac{\pi}{2}]$.
9. Simplify using $A = \frac{1}{2}ab\sin\gamma$.
10. Raise both sides to the nth power and take the logarithm.
11. WLOG $a \geq b \geq c$. Can you prove that $a + b \geq c + a \geq b + c$?
12. Order $(2x_i - x_{i+1})$ and (x_i) into decreasing sequences and take the difference of the first k terms.
13. Note that the inequality is homogeneous and WLOG let $a + b + c = 1$.
14. $\frac{1}{1+xy} = \frac{z}{z+xyz}$.

5.2 Hints to problems

1. Divide both sides by n^p.
2. Note that $\sum_{i=1}^n \ln x_i = \sum_{i=1}^n \ln \sqrt{x_i} = 0$.
3. $\frac{n}{n+\sqrt{2}} = n\frac{(1/n)}{1+(1/n)\sqrt{2}}$ and $(a_1 + \ldots + a_n)/n = 1/n$.
4. Take the inverse on both sides.
5. Write all fractions in terms of the function $f(x) = 1/x$.
6. Introduce $S = a + b + c$.
7. Transform all products and divisions using logarithms and exponentiation.
8. Rewrite the left hand side as a single sum over i containing the sum $x_1 + \ldots + x_n$.
9. Let $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$. Watch out for negative terms.